Quadratic Equations

Key Topics

- **Quadratic Equations**: A quadratic equation is an equation with a degree of two, meaning that at least one term in the equation is raised to the second power. The standard form is \( ax^2 + bx + c = 0 \) where \( a \neq 0 \).

Solving Quadratic Equations

- **Factoring Method**: This method uses the zero product property which states that if a product is zero, then at least one of its factors has to be zero.
  
  **Example**: Solve the equation \( x(x + 4) = 12 \).
  
  **Step 1**: Put the equation in standard form.
  \[ x(x + 4) = 12 \rightarrow x^2 + 4x = 12 \rightarrow x^2 + 4x - 12 = 0 \]
  
  **Step 2**: Factor the equation.
  \[ x^2 + 4x - 12 = 0 \rightarrow (x + 6)(x - 2) = 0 \]
  
  **Step 3**: If a product equals zero, one of its factors is equal to zero.
  \[ x + 6 = 0 \text{ or } x - 2 = 0 \]
  \[ x = -6 \text{ or } x = 2 \]

  So the solution set is \( \{-6, 2\} \).

- **Square Root Method**: If an expression squared is equal to a constant, then that expression is equal to the positive or negative square root of the constant.
  
  **Example**: Solve the equation \( 3(x - 1)^2 - 75 = 0 \).
  
  **Step 1**: Isolate the squared expression.
  \[ 3(x - 1)^2 - 75 = 0 \rightarrow 3(x - 1)^2 = 75 \rightarrow (x - 1)^2 = 25 \]
  
  **Step 2**: Apply the square root property.
  \[ (x - 1)^2 = 25 \rightarrow \sqrt{(x - 1)^2} = \sqrt{25} \rightarrow x - 1 = \pm \sqrt{25} \rightarrow x - 1 = \pm 5 \]
  
  **Step 3**: Solve for \( x \).
  \[ x - 1 = \pm 5 \rightarrow x = \pm 5 + 1 \]
  
  If it is \(-5\), then \( x = -5 + 1 = -4 \). If it is \( 5 \), then \( x = 5 + 1 = 6 \).

  The solution set is \( \{-4, 6\} \).

- **Completing the Square**: Not all quadratic equations can use the factoring or square root methods. Completing the square aims to transform a standard quadratic equation \( ax^2 + bx + c = 0 \) into a perfect square \( (x + A)^2 = B \).
  
  **Example**: Solve the equation \( 2x^2 - 4x + 3 = 0 \).
  
  **Step 1**: If the leading coefficient is not 1, divide by the leading coefficient.
  \[ \frac{1}{2} (2x^2 - 4x + 3) = \frac{1}{2} (0) \rightarrow x^2 - 2x + \frac{3}{2} = 0 \]
Step 2: Add the opposite of the constant term to both sides.

\[ x^2 - 2x + \frac{3}{2} - \frac{3}{2} = 0 - \frac{3}{2} \rightarrow x^2 - 2x = -\frac{3}{2} \]

Step 3: Add \( \left(\frac{b}{2}\right)^2 \) to both sides where \( b \) is the coefficient of \( x \).

\[ x^2 - 2x + \left(\frac{-2}{2}\right)^2 = -\frac{3}{2} + \left(\frac{-2}{2}\right)^2 \rightarrow x^2 - 2x + 1 = -\frac{1}{2} \]

Step 4: Write the left side of the equation as a perfect square.

\[ (x - 1)^2 = -\frac{1}{2} \]

Step 5: Apply the square root method to solve.

\[ x - 1 = \pm \sqrt{-\frac{1}{2}} \rightarrow x = 1 \pm \sqrt{-\frac{1}{2}} = 1 \pm \frac{i\sqrt{2}}{\sqrt{2}} \rightarrow x = 1 \pm \frac{i\sqrt{2}}{2} \]

The solution set is \( \left\{ 1 - \frac{i\sqrt{2}}{2}, 1 + \frac{i\sqrt{2}}{2} \right\} \)

### Quadratic Formula

- **Quadratic Formula**: For any equation of the form \( ax^2 + bx + c = 0 \) (with \( a \neq 0 \)), the solution can be found using the following formula:

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

This formula is called the **quadratic formula** and can be used to solve any quadratic equation.

**Example**: Solve the equation \( 3x^2 - 2x + 9 = 0 \).

**Step 1**: Identify \( a, b, \) and \( c \).

\[ a = 3, \quad b = -2, \quad c = 9 \]

**Step 2**: Substitute the values of \( a, b, \) and \( c \) into the quadratic equation.

\[ x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(9)}}{2(3)} \]

**Step 3**: Simplify.

\[ x = \frac{2 \pm \sqrt{4 - 108}}{6} = \frac{2 \pm \sqrt{-104}}{6} = \frac{2 \pm i\sqrt{4 \cdot 26}}{2 \cdot 3} = \frac{2 \pm 2i\sqrt{26}}{2 \cdot 3} = \frac{2(1 \pm i\sqrt{26})}{3} \]

\[ = \frac{1 \pm i\sqrt{26}}{3} \]

The solution set is \( \left\{ \frac{1 - i\sqrt{26}}{3}, \frac{1 + i\sqrt{26}}{3} \right\} \)

- **Discriminant**: The discriminant is \( b^2 - 4ac \). This is the term found inside the radical of the quadratic equation and gives information about the solutions of any equation of the form \( ax^2 + bx + c = 0 \) where \( a, b, \) and \( c \) are real numbers.
<table>
<thead>
<tr>
<th>$b^2 - 4ac$</th>
<th>Solutions (Roots)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive</td>
<td>Two distinct real roots</td>
</tr>
<tr>
<td>0</td>
<td>One real root</td>
</tr>
<tr>
<td>Negative</td>
<td>Two complex roots</td>
</tr>
</tbody>
</table>

**Example:** Check the discriminant of the example above to see if the solution set we found matches the information in the table above.

**Step 1: Identify $a$, $b$, and $c$.**

\[ a = 3, \quad b = -2, \quad c = 9 \]

**Step 2: Substitute the values of $a$, $b$, and $c$ into the discriminant.**

\[ b^2 - 4ac \rightarrow (-2)^2 - 4(3)(9) \]

**Step 3: Simplify.**

\[ (-2)^2 - 4(3)(9) = 4 - 12(9) = 4 - 108 = -104 \]

Since \( b^2 - 4ac \) simplifies to a negative for \( 3x^2 - 2x + 9 = 0 \) the table says it should have two complex roots, and we found that the solutions for \( 3x^2 - 2x + 9 = 0 \) are

\[ \left\{ \frac{1 - i\sqrt{26}}{3}, \frac{1 + i\sqrt{26}}{3} \right\} \]

which are in fact complex.

**Other Types of Equations**

### Radical Equations

- **Radical Equation:** A radical equation is an equation where the variable is inside a radical (i.e. \( \sqrt{3x - 4} = 7; \sqrt[5]{x^5 + x^2 - 7} = 9x \)).

- **Procedure for Solving Radical Equations:** The following example shows how to solve radical equations.

**Example:** Solve the equation \( \sqrt{2x - 6} = x - 3 \)

**Step 1: Square both sides of the equation.**

\[ (\sqrt{2x - 6})^2 = (x - 3)^2 \]

**Step 2: Simplify.**

\[ 2x - 6 = x^2 - 6x + 9 \]

**Step 3: Write the equation in standard form.**

\[ x^2 - 6x + 9 - 2x + 6 = 0 \]
\[ x^2 - 8x + 15 = 0 \]

**Step 4: Factor.**

\[ (x - 3)(x - 5) = 0 \]
Step 5: Use the zero product property.

\[ x = 3 \text{ or } x = 5 \]

Step 6: Check the solutions.

\[
\sqrt{2(3) - 6} = 3 - 3 \rightarrow \sqrt{6 - 6} = 0 \rightarrow \sqrt{0} = 0 \rightarrow 0 = 0 \checkmark \\
\sqrt{2(5) - 6} = 5 - 3 \rightarrow \sqrt{10 - 6} = 2 \rightarrow \sqrt{4} = 2 \rightarrow 2 = 2 \checkmark
\]

**u-Substitution**

- Equations that are higher order or that have fractional powers often can be transformed into a quadratic equation through substitution. When this is the case, we say these equations are quadratic in form.

- **Procedure for Solving Equation through u-substitution:** The following example shows how to solve equations using u-substitution.

**Example:** Solve \( x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 10 = 0 \)

**Step 1: Identify the substitution.**

\[ u = x^{\frac{1}{3}} \]

**Step 2: Transform the equation into a quadratic equation.**

\[
(u^{\frac{1}{3}})^2 - 3u^{\frac{1}{3}} - 10 = 0 \\
\Rightarrow u^2 - 3u - 10 = 0
\]

**Step 3: Solve the quadratic equation.**

\[
(u - 5)(u + 2) = 0 \\
\Rightarrow u = 5 \text{ or } u = -2
\]

**Step 4: Apply the substitution to rewrite the solution in terms of the original variable.**

\[ x^{\frac{1}{3}} = 5 \text{ or } x^{\frac{1}{3}} = -2 \]

**Step 5: Solve the resulting equation.**

\[
\left(x^{\frac{1}{3}}\right)^3 = 5^3 \rightarrow x = 125 \\
\left(x^{\frac{1}{3}}\right)^3 = (-2)^3 \rightarrow x = -8
\]

**Step 6: Check the solutions in the original equation.**

\[
(125)^{\frac{2}{3}} - 3(125)^{\frac{1}{3}} - 10 = 0 \rightarrow \sqrt[3]{(125)^2} - 3\sqrt[3]{125} - 10 = 0 \rightarrow 25 - 3(5) - 10 = 0 \rightarrow 0 = 0 \checkmark \\
(-8)^{\frac{2}{3}} - 3(-8)^{\frac{1}{3}} - 10 = 0 \rightarrow \sqrt[3]{(-8)^2} - 3\sqrt[3]{-8} - 10 = 0 \rightarrow 4 - 3(-2) - 10 = 0 \rightarrow 0 = 0 \checkmark
\]

**Factorable Equations**

- Some equations (both polynomial and with rational exponents) that are factorable can be solved using the zero product property.

- **Example with Rational Exponents:** Solve \( x^{\frac{7}{3}} - 3x^{\frac{4}{3}} - 4x^{\frac{1}{3}} = 0 \)

\[
x^{\frac{1}{3}}(x^2 - 3x - 4) = 0 \rightarrow x^{\frac{1}{3}}(x - 4)(x + 1) = 0 \\
x^{\frac{1}{3}} = 0 \rightarrow x = 0 \\
x - 4 = 0 \rightarrow x = 4
\]
\[ x + 1 = 0 \rightarrow x = -1 \]

- **Example of Polynomial:** Solve \( x^3 - 5x^2 - 9x + 45 = 0 \)

\[
(x^3 - 5x^2) + (-9x + 45) = 0 \rightarrow x^2(x - 5) + (-9)(x - 5) = 0 \rightarrow (x - 5)(x^2 - 9) = 0 \\
\rightarrow (x - 5)(x + 3)(x - 3) = 0 \\
x = 5 \text{ or } x = -3 \text{ or } x = 3 \]