Quadratic Functions

Key Definitions

- **Polynomial Function:**
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$
Where a_0 , a_1 , a_2 , … , a_n are real numbers, with $a_n \neq 0$, and n is a nonnegative integer is called a polynomial function of x with degree n. The coefficient a_n is called the leading coefficient, and a_0 is the constant.
- **Quadratic Function:** $f(x) = ax^2 + bx + c$ (a, b, and c are real numbers with $a \neq 0$)
- The graph of a quadratic function is a **parabola**.
- **The Standard Form of a Quadratic Function:** $f(x) = a(x-h)^2 + k$
- The graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola with the **vertex** located at the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Graphing Quadratic Functions

To graph the standard form: *f(x)* = a(x - h)^2 + k	
a > 0	The parabola opens up
a < 0	The parabola opens down
(h, k)	The vertex of the parabola
y-intercept	Set x = 0
Any x-intercepts	Set *f(x)* = 0

- **Example 1:** Graph the quadratic function $f(x) = -2(x-2)^2 + 2$

Since a = -2 < 0, the parabola opens down. The vertex of the parabola is (2, 2).

Find the y-intercept:

$$f(0) = -2(0-2)^2 + 2 = -2(-2)^2 + 2 = -2(4) + 2 = -8 + 2 = -6$$

The y-intercept is (0, -6).

Find any x-intercepts:

$$0 = -2(x-2)^2 + 2$$

$$0 - 2 = -2(x-2)^2 + 2 - 2 = -2(x-2)^2$$

$$\frac{-2}{-2} = \frac{-2(x-2)^2}{-2} = (x-2)^2$$

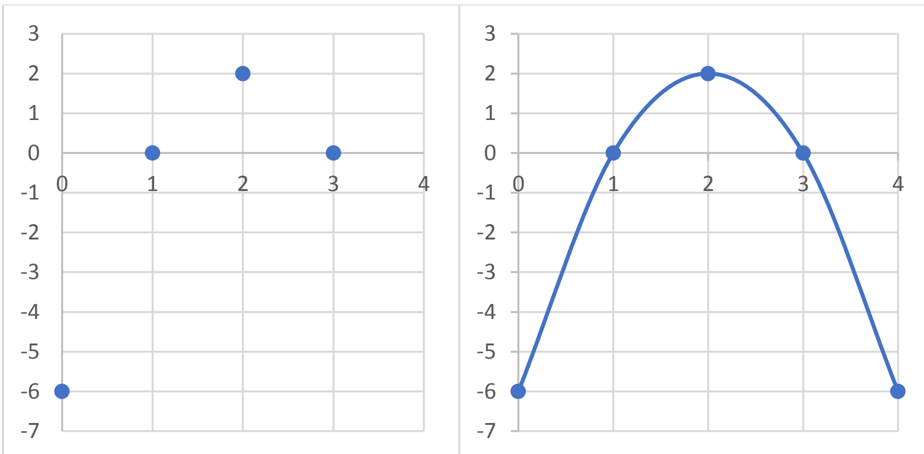
$$\sqrt{1} = \sqrt{(x-2)^2}$$

$$\pm 1 = x - 2$$

$$x = 3, 1$$

The x-intercepts are (3, 0) and (1, 0).

Plot the vertex, y-intercept, and x-intercepts. Then connect the points with a smooth curve opening down.



- When graphing quadratic functions, you need to have at least 3 points labeled on the graph. If there are no x-intercepts, you must find another point by plugging a number into the function.
- One technique for graphing a quadratic given in general form ($f(x) = ax^2 + bx + c$) is to transform the function into the standard form by completing the square.
- **Example 2**: Graph the quadratic function $f(x) = 3x^2 + 9x + 2$

First complete the square.

$$f(x) = (3x^2 + 9x) + 2 = 3(x^2 + 3x) + 2$$

$$= 3\left(x^2 + 3x + \frac{9}{4}\right) - \frac{9}{4} + 2 = 3\left(x^2 + 3x + \frac{9}{4}\right) - \frac{1}{4}$$

$$= 3\left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$$

Since a = 3 > 0, the parabola opens up. The vertex is $\left(\frac{-3}{2}, \frac{-1}{4}\right)$.

Find the y-intercept (use the original form):

$$f(0) = 3(0)^2 + 9(0) + 2 = 0 + 0 + 2 = 2$$

The y-intercept is (0, 2).

Find any x-intercepts (use the standard form):

$$0 + \frac{1}{4} = 3\left(x + \frac{3}{2}\right)^2 - \frac{1}{4} + \frac{1}{4}$$

$$\frac{\frac{1}{4}}{3} = \frac{3\left(x+\frac{3}{2}\right)^2}{3} = \left(x+\frac{3}{2}\right)^2$$

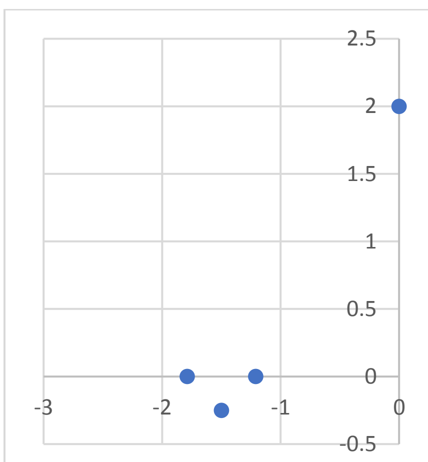
$$\sqrt{\frac{1}{12}} = \sqrt{\left(x+\frac{3}{2}\right)^2}$$

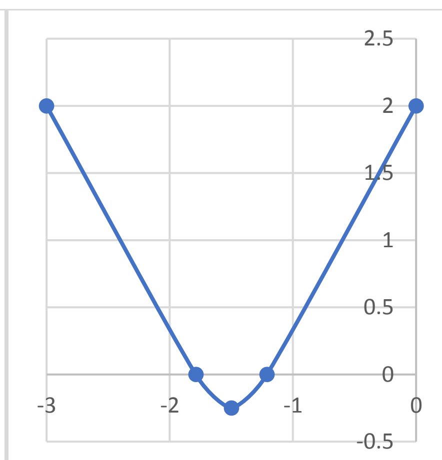
$$-\frac{3}{2} \pm \sqrt{\frac{1}{12}} = x + \frac{3}{2} - \frac{3}{2}$$

$$-\frac{3}{2} \pm \sqrt{\frac{1}{12}} = x$$

The x-intercepts are $(-\frac{3}{2} + \sqrt{\frac{1}{12}}, 0)$ and $(-\frac{3}{2} - \sqrt{\frac{1}{12}}, 0)$

Plot the vertex, y-intercept, and x-intercepts. Then connect the points with a smooth curve opening up.





- **Example 3**: Graph the quadratic function $f(x) = -2x^2 + 8x + 3$

First find the vertex.

$$x = -\frac{b}{2a} = -\frac{8}{2(-2)} = \frac{-8}{-4} = 2$$

$$f(2) = -2(2)^2 + 8(2) + 3 = -8 + 16 + 3 = 11$$

The vertex is (2, 11).

The parabola opens down since $a = -2$.

Find additional points near the vertex.

Use $x = 0, 1, 3, 4$

Find $f(0), f(1), f(3), f(4)$.

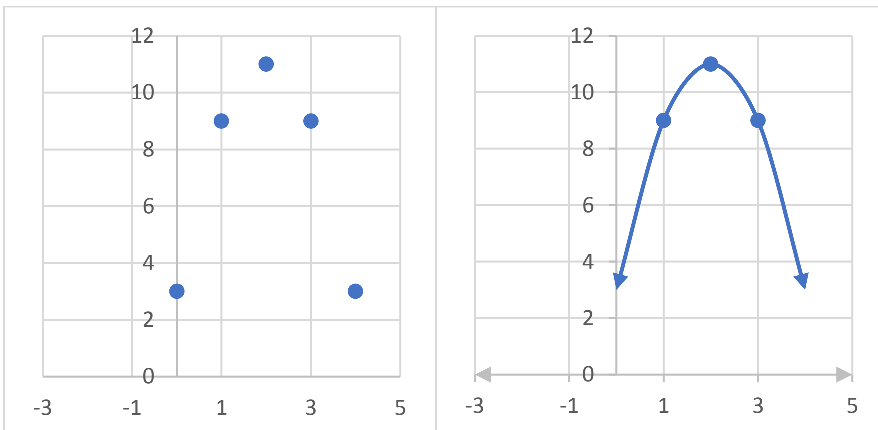
$$f(0) = -2(0)^2 + 8(0) + 3 = 0 + 0 + 3 = 3$$

$$f(1) = -2(1)^2 + 8(1) + 3 = -2 + 8 + 3 = 9$$

$$f(3) = -2(3)^2 + 8(3) + 3 = -18 + 24 + 3 = 9$$

$$f(4) = -2(4)^2 + 8(4) + 3 = -32 + 32 + 3 = 3$$

Plot the vertex and additional points. Then connect the points with a smooth curve opening down.



Finding the Equation of the Parabola

- **Example 1:** Find the equation of a parabola whose graph has a vertex at (2, 5) and that passes through the point (6, -27). Express the quadratic function in both standard and general forms.

  Use the standard form. Since the vertex is (2, 5), h = 2 and k = 5.

$$f(x) = a(x - h)^2 + k$$
$$f(x) = a(x - 2)^2 + 5$$

  Use the point (6, -27) to find a.

$$-27 = a(6 - 2)^2 + 5 = 16a + 5$$
$$-27 = 16a + 5$$
$$-32 = 16a$$
$$-2 = a$$

  Rewrite in standard form.

$$f(x) = -2(x - 2)^2 + 5$$

  Find the general form by simplifying.

$$f(x) = -2(x - 2)^2 + 5 = -2(x^2 - 4x + 4) + 5 = -2x^2 + 8x - 8 + 5 = -2x^2 + 8x - 3$$
$$f(x) = -2x^2 + 8x - 3$$

Applications

- **Example 1: Path of a Punted Football:** The path of a particular punt follows the quadratic function: $h(x) = -\frac{1}{8}(x-5)^2 + 50$, where *h(x)* is the height of the ball in yards and *x* corresponds to the horizontal distance in yards. Assume $x = 0$ corresponds to midfield.
  First, find the maximum height the ball achieves.
  Then, find the horizontal distance the ball covers. Assume the height is zero when the ball is kicked and when the ball is caught.
  To find the maximum height the ball achieves, consider what kind of quadratic this punt would resemble. Since $a = -\frac{1}{8}$ which means $a < 0$, the parabola opens down. Since the parabola opens down, the vertex will be the highest point on the parabola and will also correspond with the maximum height of the football.
  To find the vertex, consider the standard form of a quadratic function: $f(x) = a(x-h)^2 + k$. The vertex is (h, k). In this case, the vertex is $(5{,}50)$. Since the y-value (*h(x)*) is 50, the height the ball achieves is 50 yards.
  To find the horizontal distance the ball covers, remember that *h(x)* is the height and the ball has a height of 0 when it is kicked and when it is caught. To find the horizontal distance, set *h(x)* equal to 0.
  $$h(x) = -\frac{1}{8}(x-5)^2 + 50$$
  $$0 = -\frac{1}{8}(x-5)^2 + 50$$
  Add $\frac{1}{8}(x-5)^2$ to both sides.
  $$\frac{1}{8}(x-5)^2 = 50$$
  $$\frac{8}{1} \cdot \frac{1}{8}(x-5)^2 = 50 \cdot \frac{8}{1}$$
  $$(x-5)^2 = 400$$
  $$\sqrt{(x-5)^2} = \sqrt{400}$$
  $$(x-5) = \pm 20$$
  $$x = \pm 20 + 5$$
  $$x = 25, -15$$
  The ball was kicked at -15 and caught at 25. Find the distance between these two points.
  $$|25 - (-15)| = 40 \text{ yards}$$

Polynomial Functions of Higher Degree

Key Definitions

- Let n be a nonnegative integer and let $a_n, a_{n-1}, \dots, a_1, a_0$ be real numbers with $a_n \neq 0$. The function
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$$
is called a **polynomial function of *x* with degree *n***. The coefficient a_n is called the leading coefficient.
- Let *n* be a positive integer and the coefficient $a \neq 0$ be a real number. The function $f(x) = ax^n$ is called a **power function of degree n**.
- Intermediate value theorem: Let a and b be real numbers such that $a < b$ and let f be a polynomial function. If $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero between a and b.
- When a zero or root appears more than once, it is called a **repeated root**. The number of times that a zero repeats is called its **multiplicity**.
- If $(x-a)^n$ is a factor of a polynomial f, then a is called a **zero of multiplicity *n*** of f.
- End behavior
- Do a chart or two

Identifying Polynomials and Their Degree

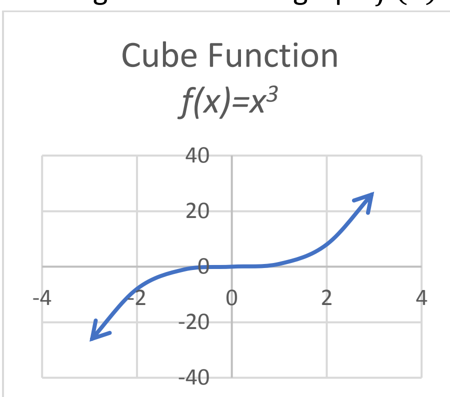
- The powers or exponents in a polynomial must be nonnegative integers.
- The degree of the polynomial is the greatest power or exponent in the polynomial.
- **Example**: Determine whether the function is a polynomial function. If it is a polynomial function, state the degree of the polynomial.
- **(a)** $f(x) = 12 - 13x^5$
This function is a polynomial with degree 5.
- **(b)** $g(x) = 8$
This function is a polynomial with degree 0 since $g(x)$ can be rewritten as $8x^0$.
- **(c)** $h(x) = x^6(3x+5)^2$
This function is a polynomial with degree 8 since $x^6(3x+5)^2 = 9x^8 + 30x^7 + 25x^6$.
- **(d)** $F(x) = 2\sqrt{x} + 3$
This function is not a polynomial since $\sqrt{x} = x^{1/2}$. The power needs to be a nonnegative integer.
- **(e)** $G(x) = 4x^2 - 5x^{-3}$
This function is not a polynomial since it has a negative exponent.
- **(f)** $H(x) = 3x^4 + 2x^3 - \frac{5}{x^4}$

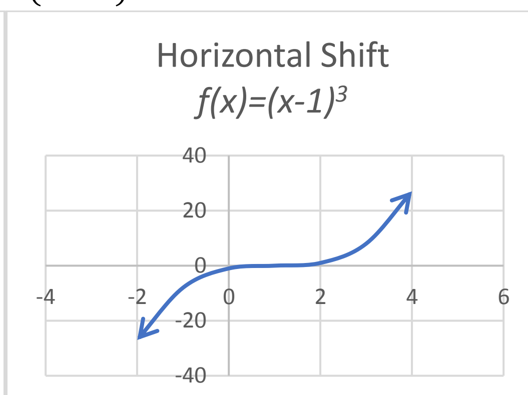
This function is not a polynomial since it has a negative exponent ($\frac{5}{x^4} = 5x^{-4}$).

Graphing Polynomial Functions Using Transformations

Characteristics of Power Functions: $f(x) = x^n$		
	n even	**n odd**
Symmetry	y-axis	Origin
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$
Range	$[0, \infty)$	$(-\infty, \infty)$
Some key points that lie on the graph	$(-1,1), (0,0)$, and $(1,1)$	$(-1,-1), (0,0)$, and $(1,1)$
Increasing	$(0, \infty)$	$(-\infty, \infty)$
Decreasing	$(-\infty, 0)$	N/A

- Example: Graph the function $f(x) = (x-1)^3$.
  Start with $f(x) = x^3$.
  Shift it to the right one unit to graph $f(x) = (x-1)^3$.





Identifying Real Zeros of a Polynomial Function

- To find real zeros, set the function equal to zero and solve for x.
- **Example 1:** Find the zeros of the polynomial function: $f(x) = x^3 + 12x^2 + 35x$
  First, set the function equal to 0.
  $$0 = x^3 + 12x^2 + 35x = x(x^2 + 12x + 35) = x(x+7)(x+5)$$
  $$0 = x \;\; or \;\; 0 = x + 7 \;\; or \;\; 0 = x + 5$$
  Since $x = 0, -7, -5$, the zeros are 0, -7, and -5
- **Example 2:** Find the zeros of the polynomial function and state their multiplicities: $f(x) = (x-1)^3(x+\frac{2}{3})^2(x+9)$
  Set the function equal to 0.
  $$0 = (x-1)^3(x+\frac{2}{3})^2(x+9)$$
  $$0 = (x-1)^3 \;\; or \;\; 0 = (x+\frac{2}{3})^2 \;\; or \;\; 0 = (x+9)$$

$$x = 1, -\frac{2}{3}, -9$$

Since $(x-1)^3$ has a power of 3, $x = 1$ has a multiplicity of 3.
$x = -\frac{2}{3}$ has a multiplicity of 2.
$x = -9$ has a multiplicity of 1.

- **Example 3:** Find a polynomial of degree 6 whose zeros are:
  -8 (multiplicity 2), 0 (multiplicity 3), 5 (multiplicity 1)
  If $x = a$ is a zero, then $(x - a)$ is a factor.

$$f(x) = (x - -8)^2(x-0)^3(x-1)^1 = (x+8)^2(x)^3(x-1)$$
$$f(x) = x^3(x^2 + 16x + 64)(x-1)$$
$$f(x) = x^3(x^3 + 15x^2 + 48x - 64)$$
$$f(x) = x^6 + 15x^5 + 48x^4 - 64x^3$$

Graphing General Polynomial Functions

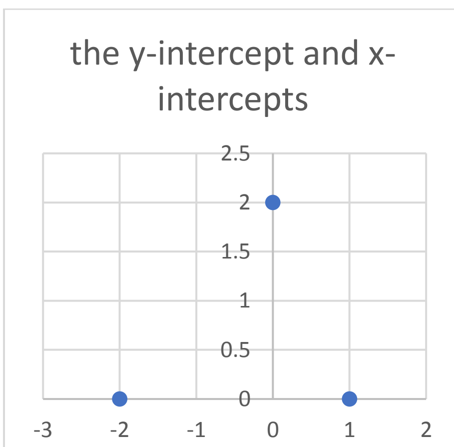
- To graph a polynomial function, find the y-intercepts and x-intercepts and plot them. Then divide the domain into intervals where the polynomial is positive or negative to find points in those intervals to help sketch the graph. (Sometimes, there are no x-intercepts.)
- **Example 1:** Sketch the graph of $f(x) = (x+2)(x-1)^2$.
  Find the y-intercept by setting x equal to 0.

$$f(0) = (0+2)(0-1)^2 = (2)(-1)^2 = 2$$

(0,2) is the y-intercept
Find any x-intercepts by setting the function equal to 0.

$$0 = (x+2)(x-1)^2$$
$$x = -2, 1$$

(-2, 0) and (1, 0) are the x-intercepts.
Plot the intercepts.



Divide the x-axis into intervals based on the x-intercepts.

$$(-\infty, -2), (-2, 1), (1, \infty)$$

Select a number in each interval and test each interval. The function either crosses the x-axis at an x-intercept or touches the x-axis at an x-intercept. Therefore, we need to check if the points are positive or negative.

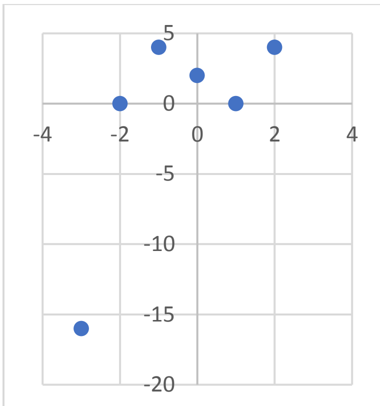
In this case, we check $x = -3, -1, 2$

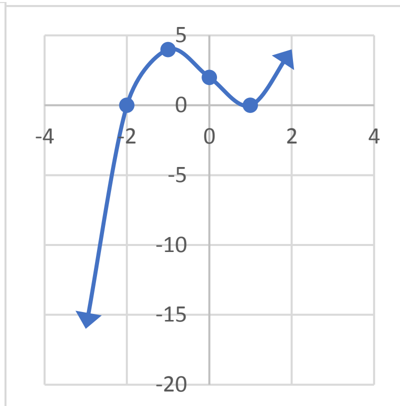
$$f(-3) = (-3+2)(-3-1)^2 = (-1)(-4)^2 = -16$$
$$f(-1) = (-1+2)(-1-1)^2 = (1)(-2)^2 = 4$$
$$f(2) = (2+2)(2-1)^2 = (4)(1)^2 = 4$$

For $x = -3$, the graph goes below the x-axis because $f(-3)$ is negative.

The other two are positive and stay above the x-axis.

Graph the new points and then sketch the curve.





- **Example 2:** Sketch the graph of $f(x) = 2x^4 - 8x^2$.

  Find the y-intercept by setting x equal to 0.

  $$f(0) = 0$$

  The y-intercept is the point (0, 0).

  Then find the zeros of the polynomial:

  Factor the polynomial.

  $$f(x) = 2x^4 - 8x^2 = (2x^2)(x^2 - 4) = 2x^2(x-2)(x+2)$$

  Set the polynomial equal to 0.

  $$0 = 2x^2(x-2)(x+2)$$

  0 is a zero with multiplicity 2. Since the multiplicity is even, the graph will touch the x-axis and then turn around.

  2 is a zero with multiplicity 1. Since the multiplicity is odd, the graph crosses the x-axis at this point.

  -2 is a zero with multiplicity 1. Since the multiplicity is odd, the graph crosses the x-axis at this point.

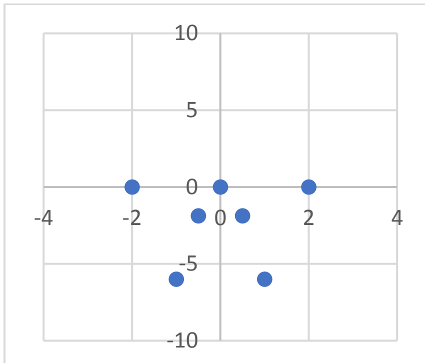
  Determine the end behavior: $f(x) = 2x^4 - 8x^2$ will behave like $f(x) = 2x^4$ which has an even degree and positive coefficient, so the graph will have end behavior similar to the square function where the two ends point in the same direction (because the degree is even) and they will both point up (because the coefficient is positive).

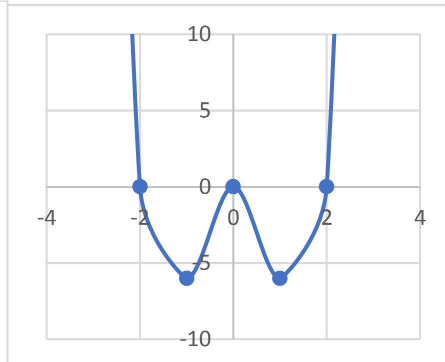
  Find additional points.

  Plug in different x values to find other points. (Try $x = -1, -\frac{1}{2}, \frac{1}{2}, 1$.)

$$f(-1) = -6$$
$$f\left(-\frac{1}{2}\right) = -\frac{15}{18}$$
$$f\left(\frac{1}{2}\right) = -\frac{15}{18}$$
$$f(1) = -6$$

Then graph the points and use the multiplicities and end behavior to sketch the graph.





Relation of Multiplicity of a Zero to a Graph If a is a zero of *f(x)*, then		
Multiplicity of a	F(x) on either side of x = a	Graph of function at the intercept
Even	Does not change sign	Touches the x-axis (turns around) at point (a, 0)
Odd	Changes sign	Crosses the x-axis at point (a, 0)

Dividing Polynomials

Long Division of Polynomials

- **Example 1:** Divide $3x^4 + 2x^3 + x^2 + 4$ by $x^2 + 1$.
  In both polynomials $0x$ needs to be inserted as a placeholder because neither of them has an x-term.

  Start by **dividing** $3x^4$ by x^2 ($\frac{3x^4}{x^2} = 3x^2$).
  Then **multiply** $3x^2$ and $x^2 + 0x + 1$ (answer: $3x^4 + 0x^3 + 3x^2$).
  Then **subtract** $3x^4 + 0x^3 + 3x^2$ from $3x^4 + 2x^3 + x^2$. Then **bring down** the next place ($0x$). Continue the process until there are no more places to bring down.
  In this case, the remainder is $-2x + 6$.

$$
\begin{array}{r}
3x^2+2x-2 \\
x^2+0x+1\overline{\big)\,3x^4+2x^3+x^2+0x+4} \\
\underline{3x^4+0x^3+3x^2} \\
0x^4+2x^3-2x^2+0x \\
\underline{0x^4+2x^3+0x^2+2x} \\
0x^4+0x^3-2x^2-2x+4 \\
\underline{0x^4+0x^3-2x^2+0x-2} \\
0x^4+0x^3+0x^2-2x+6
\end{array}
$$

$$\frac{3x^4+2x^3+x^2+4}{x^2+1}=3x^2+2x-2+\frac{-2x+6}{x^2+1}$$

Synthetic Division of Polynomials

- When the divisor is a linear factor of the form $x-a$ or $x+a$, **synthetic division** can be used to divide polynomials.
- If $x-a$ is the divisor, then a is the number used in synthetic division.
- If any terms in the polynomial are missing, use 0 as a placeholder.
- **Example 1:** Divide $3x^5-2x^3+x^2-7$ by $x+2$.
  Use -2 to divide because of $x+2$.
  Rewrite $3x^5-2x^3+x^2-7$ with placeholders: $3x^5+0x^4-2x^3+x^2+0x-7.$
  Bring down the first number (3).
  Multiply -2 times 3 (answer: -6) and put it in the second column to the right of the 3.
  Then **add** the numbers in the second column $(0+-6=-6)$ and put it at the bottom of the column.
  Then **multiply** -2 times -6 (answer: 12) and put it in the third column to the right of the -6.
  Then **add** the numbers in the third column $(-2+12=10)$ and put it at the bottom of the column.
  Continue to multiply and add until all the columns are filled.

-2	3	0	-2	1	0	-7
		-6	12	-20	38	-76
	3	-6	10	-19	38	-83

The last column will have the remainder (-83).
Rewrite the new numbers (coefficients) with variables starting with one less power than the original polynomial. In this case, start with x^4.

$$3x^4-6x^3+10x^2-19x+38-\frac{83}{x+2}$$