Linear Inequalities

Key Definitions

- **Linear Inequality:** A linear inequality is a linear relation that uses one of the four inequality symbols instead of the equals sign used in linear equations.
  Example: $9x + 2 \geq 11$ *is a linear inequality*
- **Union:** The union of two sets ($A \cup B$) is the set formed by combining all the elements from both sets.
  Example: $A = \{George, Samuel, Yvette\}$ $\quad B = \{Ellen, Ryan, Yvette\}$
  $A \cup B = \{Ellen, George, Samuel, Ryan, Yvette\}$
- **Intersection:** The intersection of two sets ($A \cap B$) is the set formed by the elements that are in both sets.
  Example: $A = \{George, Samuel, Yvette\}$ $\quad B = \{Ellen, Ryan, Yvette\}$
  $A \cup B = \{Yvette\}$

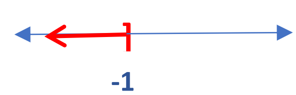
Solving Linear Inequalities

- Linear inequalities are solved like linear equations. A linear equation gives you an exact solution that makes the equation true such as $x = 3$, but a linear inequality gives you an interval or range of numbers that make the inequality true such as $x \leq 3$.
- **Example:** Solve the following equation and inequality.

$$\begin{aligned} 5 &= 3 - 2x \\ 5 - 3 &= -2x \\ 2 &= -2x \\ \frac{2}{-2} &= \frac{-2x}{-2} \\ -1 &= x \\ x &= -1 \end{aligned}$$

$$\begin{aligned} 5 &\leq 3 - 2x \\ 5 - 3 &\leq -2x \\ 2 &\leq -2x \\ \frac{2}{-2} &\leq \frac{-2x}{-2} \\ -1 &\geq x \\ x &\leq -1 \end{aligned}$$

NOTE: When multiplying or dividing an inequality by a negative number the inequality sign switches. Also notice how the equation gives only one solution, $x = -1$, and the inequality gives a range of solutions which can be written as any of the following:

Inequality Notation	Set Notation	Interval Notation	Number Line
$\boldsymbol{x \leq -1}$	$\{x \mid x \leq -1\}$	$(-\infty, -1]$	

Solving Double Linear Inequalities

- Double linear inequalities are solved in the same way as a single linear inequality. The main thing to remember is if you apply one operation to a part of the inequality then you have to apply it to each part. This is the same thing as equations. If you add 2 to the left side, then you must add 2 to the right side as well.
- **Example:** Solve the following double linear inequality: $2x + 1 < 4x + 2 < 2x + 5$

$$2x + 1 < 4x + 2 < 2x + 5$$
$$2x - 2x + 1 < 4x - 2x + 2 < 2x - 2x + 5$$
$$1 < 2x + 2 < 5$$
$$1 - 2 < 2x + 2 - 2 < 5 - 2$$
$$-1 < 2x < 3$$
$$\frac{-1}{2} < \frac{2x}{2} < \frac{3}{2}$$
$$-\frac{1}{2} < x < \frac{3}{2}$$

Symbol Guide

Algebra Review Symbols		
Term	Symbol	Use
Less than	$<$	Identifies the quantity to the left of the symbol as being **less than** the quantity to the right of the symbol
Greater than	$>$	Identifies the quantity to the left of the symbol as being **greater than** the quantity to the right of the symbol
Less than or equal to	$\leq$	Identifies the quantity to the left of the symbol as being **less than or equal to** the quantity to the right of the symbol
Greater than or equal to	$\geq$	Identifies the quantity to the left of the symbol as being **greater than or equal to** the quantity to the right of the symbol
Union	$\cup$	Combines the elements of two sets into one set
Intersection	$\cap$	Creates a set of the common elements from two sets

Polynomial and Rational Inequalities

Key Definitions

- **Polynomial Inequalities:** A polynomial inequality has a degree of 2 or higher.
- **Zeros:** The zeros of a polynomial are the values of x that make the polynomial equal to zero.
- **Test Intervals:** The zeros of a polynomial divide the real number line into test intervals where the value of the polynomial is either positive or negative.

Solving Polynomial Inequalities

- **Example:** Solve the following quadratic inequality: $x^2 - 5x \leq 6$

Step 1: Write the inequality in standard form and factor the left side.

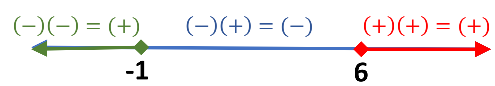
$$x^2 - 5x - 6 \leq 0$$
$$(x-6)(x+1) \leq 0$$

Step 2: Identify zeros.

$$(x-6)(x+1) = 0$$
$$x = 6 \text{ or } x = -1$$

Step 3: Draw the number line with zeros labeled.

Step 4: Determine the sign of the polynomial in each interval by testing a value of x from that interval.



$x = -2$	$x = 0$	$x = 7$
$(-2-6)(-2+1)$	$(0-6)(0+1)$	$(7-6)(7+1)$
$= (-8)(-1)$	$= (-6)(1)$	$= (1)(8)$
$= 8$	$= -6$	$= 7$

Step 5: Identify which interval(s) make the inequality true.

The intervals where the value is negative make this inequality true as $(x-6)(x+1)$ must be equal to or less than zero so here it is the interval between -1 and 6.

Step 6: Write the solution in interval notation.

$$[-1, 6]$$

NOTE: Polynomial inequalities with higher degrees will have more test intervals as there are more zeros to consider.

Solving Rational Inequalities

- **Example:** Solve the following rational inequality: $\frac{x-3}{x^2-25} \geq 0$

Step 1: Factor the numerator and denominator if possible.

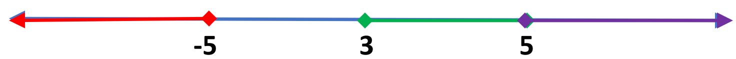
$$\frac{x-3}{(x+5)(x-5)} \geq 0$$

Step 2: Identify the zeros of the numerator and denominator and state the domain restrictions (aka the values that make the denominator zero).

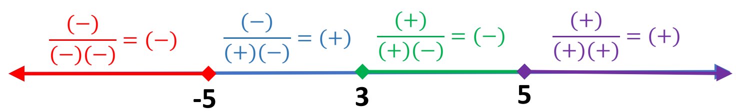
Zeros: $x = 3, x = -5, x = 5$

Domain Restrictions: $x \neq -5, x \neq 5$

Step 3: Draw the number line with zeros labeled.



Step 4: Test the intervals.



$x = -6$

$$\frac{(-6-3)}{(-6+5)(-6-5)} = \frac{(-9)}{(-1)(-11)} = -\frac{9}{11}$$

$x = 0$

$$\frac{(0-3)}{(0+5)(0-5)} = \frac{(-3)}{(5)(-5)} = \frac{3}{25}$$

$x = 4$

$$= \frac{(4-3)}{(4+5)(4-5)} = \frac{(1)}{(9)(-1)} = -\frac{1}{9}$$

$x = 6$

$$\frac{(6-3)}{(6+5)(6-5)} = \frac{(3)}{(11)(1)} = \frac{3}{11}$$

Step 5: Identify which interval(s) make the inequality true.

The intervals where the value is positive make this inequality true as $\frac{x-3}{(x+5)(x-5)}$ must be equal to or greater than zero so here it is the intervals $(-5, 3]$ and $(5, \infty)$.

Step 6: Write the solution in interval notation.

$$(-5, 3] \cup (5, \infty)$$

Notice that we use a bracket for the 3 and parentheses for the -5 and 5 as we stated earlier that we could not use -5 or 5 in the domain.

Absolute Value Equations and Inequalities

Absolute Value

- **<u>Definition</u>:** The absolute value of a real number a, denoted by the symbol $|a|$, is defined by:

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

- **<u>Properties of Absolute Value</u>:** For all real numbers a and b,
  - $|a| \geq 0$ (the absolute value of any number is positive or zero).
  - $|-a| = |a|$

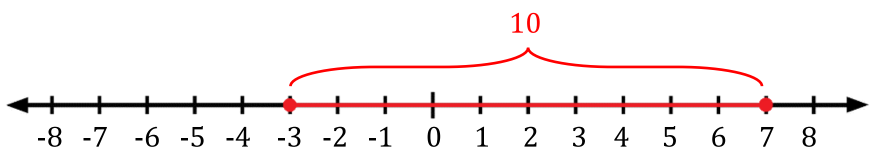
    Example: $|-7| = |7| = 7$
  - $|ab| = |a| \cdot |b| = |a||b|$

    Example: $|3x| = |3||x| = 3x$
  - $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ where $b \neq 0$

    Example: $\left|\frac{-3}{4}\right| = \frac{|-3|}{|4|} = \frac{3}{4}$
- **<u>Distance Between Two Points on a Number Line</u>:** If a and b are real numbers, the distance between a and b is the absolute value of their difference given by $|a - b|$ or $|b - a|$.

  Example: Find the difference between -3 and 7 on the real number line.

$$|7 - (-3)| = |7 + 3| = |10| = 10$$



Note: The order of a and b inside the absolute value bars does not matter, so $|a - b| = |b - a|$ regardless of what a and b are.

Equations and Inequalities with Absolute Values

- **<u>Absolute Value Equation</u>:** If $|x| = a$, then $x = -a$ or $x = a$, where $a \geq 0$

  Example: Solve $|5x - 1| = 9$.

  If the absolute value of an expression is 9, then that expression is -9 or 9

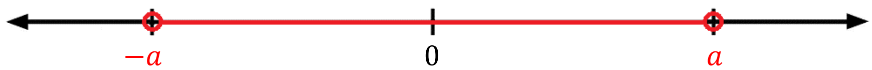
  $5x - 1 = 9$ or $5x - 1 = -9$

  $5x = 10$ $\quad$ $5x = -8$

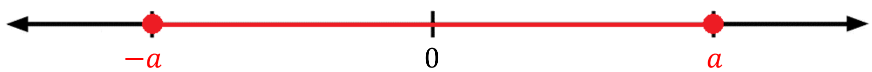
  $x = 2$ $\quad$ $x = \frac{-8}{5}$

The solution set is $\left\{2, \frac{-8}{5}\right\}$

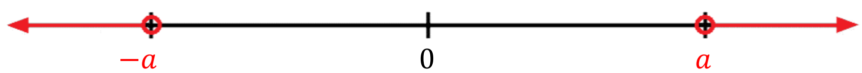
- **<u>Properties of Absolute Value Inequalities</u>:**
  - $|x| < a$ is equivalent to $-a < x < a$



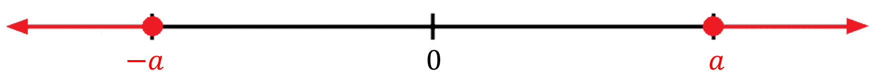
  - $|x| \leq a$ is equivalent to $-a \leq x \leq a$



  - $|x| > a$ is equivalent to $x < -a$ or $x > a$



  - $|x| \geq a$ is equivalent to $x \leq -a$ or $x \geq a$



Note: $a > 0$