Exponential Functions

An exponential function with base \( b \) is denoted by \( f(x) = b^x \) where \( b \) and \( x \) are any real numbers such that \( b > 0 \) and \( b \neq 1 \). Review sections 0.2-0.3 for properties of exponents.

Example 1: Let \( f(x) = 4^x \), \( h(x) = \frac{1}{9}^x \), \( g(x) = 10^{x-1} \). Find the following values. If an approximation is required, approximate to four decimal places.

\[
\begin{align*}
    f(0) &= 4^0 = 1 \\
    f(\pi) &= 4^\pi \approx 77.8802 \\
    f\left(-\frac{3}{2}\right) &= 4^{-1.5} = \frac{1}{32} \\
    g(2.3) &= 10^{2.3-1} = 10^{1.3} \approx 19.9526 \\
    h(0) &= 9^{-1} = 0.1111 \\
    h\left(-\frac{3}{2}\right) &= \frac{1}{9}^{3/2} = \frac{1}{27} \\
\end{align*}
\]

Graphs of Exponential Functions

Exponential functions can be graphed by plotting points. Usually, it is useful to find the points for \( x = 0, 1, -1 \).

Example 1: Graph the function \( f(x) = 4^x \).
Label the y-intercept by finding \( f(x) = 4^x \) when \( x = 0 \).
\[
    f(0) = 4^0 = 1
\]
First point is (0, 1).
Then find \( f(1) \) and \( f(-1) \).
\[
    f(1) = 4^1 = 4 \\
    f(-1) = 4^{-1} = 0.25
\]
Plot the points and then sketch the curve with a horizontal asymptote.

\[
y = 4^x
\]

- **Example 2:** Graph the function \( f(x) = 2^{x+1} - 2 \). State the domain and range of the function.

  First, identify the base function: \( f(x) = 2^x \)

  Identify the base function y-intercept and horizontal asymptote: (0, 1) and \( y = 0 \).

  Since a 1 is added to \( x \), the graph is shifted one unit to the left.

  Since a 2 is subtracted from \( 2^{x+1} \), the graph is shifted two units down.

  Shift the y-intercept from (0, 1) to (0-1, 1-2) = (-1, -1).

  Shift the horizontal asymptote down two units from \( y = 0 \) to \( y = -2 \).

  Find additional points on the graph. \( f(0) = 2^{0+1} - 2 = 2 - 2 = 0 \)

  \( f(1) = 2^{1+1} - 2 = 4 - 2 = 2 \)

  Plot the points and sketch the graph with a smooth curve.

  The domain of the function is \( (-\infty, \infty) \).

  The range of the function is \( (-2, \infty) \).

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**The Natural Base e**

- The irrational number \( e \) appears in many applications and is called the natural base. The exponential function with base \( e \) \( f(x) = e^x \) is called the exponential function or the natural exponential function.
• \( e \approx 2.71828 \)

![Graph of \( f(x) = e^x \)]

• **Example 1:** Graph the function \( f(x) = 3 + e^{2x} \).
  First, create a table with points to plot on the graph.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 3 + e^{2x} )</th>
<th>((x, y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>3.02</td>
<td>(-2, 3.02)</td>
</tr>
<tr>
<td>-1</td>
<td>3.14</td>
<td>(-1, 3.14)</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
<td>(0, 4)</td>
</tr>
<tr>
<td>1</td>
<td>10.39</td>
<td>(1, 10.39)</td>
</tr>
<tr>
<td>2</td>
<td>57.60</td>
<td>(2, 57.60)</td>
</tr>
</tbody>
</table>

Note: These values need to be found using a calculator and will need to be rounded.

### Applications

• Exponential functions describe either growth or decay.

• **Example 1:** Doubling Time of Populations
  
  **Use the doubling time growth model:**
  
  \[ P = P_0 2^{t/d} \]

  \( P \) is the population at time \( t \).
  \( P_0 \) is the population at time \( t = 0 \).
  \( d \) is the doubling time.

  The current population of an island is 800,000, and the population is expected to double in 20 years. Estimate the population in 4 years. Round your answer to the nearest thousand.

  \[ P_0 = 800000, \quad t = 4, \quad d = 20 \]

  \[ P = P_0 2^{t/d} = (800000) \left(2^{\frac{4}{20}}\right) = (800000)(2^{\frac{1}{5}}) \approx 918959 \]

  In 4 years, there will be approximately 919,000 people on the island.

• **Example 2:** Radioactive Decay: half-life
Use the half-life model:

\[ A = A_0 \left( \frac{1}{2} \right)^{t/h} \]

The radioactive isotope of potassium which is used in the diagnosis of brain tumors, has a half-life of 12.36 hours. If 700 milligrams of this potassium are taken, how many milligrams will remain after 48 hours? Round to the nearest milligram.

\[ A_0 = 700, \ t = 48, \ h = 12.36 \]

\[ A = 700 \left( \frac{1}{2} \right)^{48/12.36} \approx 700(0.0678) \approx 47.46 \]

After 48 hours, there are approximately 47 milligrams of potassium left.

- **Example 3**: Compound Interest

If a principal \( P \) is invested at an annual rate \( r \) compounded \( n \) times a year, then the amount \( A \) in the account at the end of \( t \) years is given by

\[ A = P \left(1 + \frac{r}{n}\right)^{nt} \]

The annual interest rate \( r \) is expressed as a decimal.

<table>
<thead>
<tr>
<th>Typical Number of Times Interest Is Compounded</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annually</td>
<td>1</td>
</tr>
<tr>
<td>Semiannually</td>
<td>2</td>
</tr>
<tr>
<td>Quarterly</td>
<td>4</td>
</tr>
<tr>
<td>Monthly</td>
<td>12</td>
</tr>
<tr>
<td>Weekly</td>
<td>52</td>
</tr>
<tr>
<td>Daily</td>
<td>356</td>
</tr>
</tbody>
</table>

If $4500 is deposited in an account paying 4% compounded monthly, how much will you have in the account in 8 years?

\[ P = 4500, \ r = 0.04, \ n = 12, \ t = 8 \]

\[ A = 4500(1 + \frac{0.04}{12})^{12\times8} = 4500(1 + \frac{0.04}{12})^{96} \approx 6193.78 \]

You will have approximately $6193.78 in the account.

- **Example 4**: Continuous Compound Interest

If a principal \( P \) is invested at an annual rate \( r \) compounded continuously, then the amount \( A \) in the account at the end of \( t \) years is given by

\[ A = Pe^{rt} \]

The annual interest rate \( r \) is expressed as a decimal.

If $5000 is deposited in a savings account paying 3.5% a year compounded continuously, how much will you have in the account in 8 years?

\[ P = 5000, \ r = 0.035, \ t = 8 \]

\[ A = (5000)e^{0.035\times8} = (5000)e^{0.28} \approx 6615.65 \]

There will be $6615.65 in the account in 8 years.
Logarithmic Functions

- For $x > 0$, $b > 0$, and $b \neq 1$, the logarithmic function with base $b$ is denoted by $f(x) = \log_b x$ where

$$y = \log_b x \text{ if and only if } x = b^y$$

Read “log base b of x”

- **Example 1**: Rewrite the following logarithms in exponential form using $y = \log_b x$ if and only if $x = b^y$

  Where $b$, the base, is represented in green, $x$, the information within our logarithm and the solution in our exponential, is represented in blue, and $y$, the solution to our logarithm and the exponent in our exponential is represented in pink.

  a) $\log_3 9 = 2$

  Exponential form: $9 = 3^2$

  b) $\log_5 125 = 3$

  Exponential form: $125 = 5^3$

  c) $\log_{16} 8 = \frac{1}{2}$

  Exponential form: $8 = 16^{\frac{1}{2}}$

- **Example 2**: Rewrite the following exponentials in logarithmic form using $y = \log_b x$ if and only if $x = b^y$

  Where $b$, the base, is represented in green, $x$, the information within our logarithm and the solution in our exponential, is represented in blue, and $y$, the solution to our logarithm and the exponent in our exponential is represented in pink.

  a) $216 = 6^3$

  Logarithmic form: $\log_3 9 = 2$

  b) $125 = 5^3$

  Logarithmic form: $\log_3 9 = 2$

  c) $125 = 5^3$

  Logarithmic form: $\log_3 9 = 2$

- **Example 3**: Evaluate the exact value of the following logarithm:

$$\log_2 8 = ?$$
Step 1: Figure out the base of the exponent.

\[ \log_2 8 = ? \]

Our base in this problem is 2

Step 2: Ask yourself “2 to what power will give me 8?”

We know that 2 to the power of 3 is 8

Step 3: Change the logarithm into exponential form

\[ 2^3 = 8 \]

### Common and Natural Logarithms

- The bases 10 and e are 2 of the most common logarithmic functions. Because they are common, we rewrite the logarithm in a simpler way.
  - Instead of \( \log_{10} x \), you will most likely see it written as \( \log x \). In other words, if there is no base written on your logarithm, you may assume it is base 10
  - Instead of \( \log_e x \), you will most likely see it written as \( \ln x \). In other words, \( \ln \) is another way to write a logarithm with base e.

### Graphs of Logarithmic Functions

- Below is the graph of \( f(x) = \log x \)

![Graph of \( y = \log x \)](image)

- **Interpreting the graph**: To begin interpreting the graph, let’s take a look at a few major points.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>Importance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>This tells us that ( f(x) ) has an x-intercept at (1,0)</td>
</tr>
<tr>
<td>1000</td>
<td>3</td>
<td>This shows that as ( x ) gets further away from its x-intercept, the y-values increase slowly.</td>
</tr>
<tr>
<td>( \frac{1}{1000} )</td>
<td>-100</td>
<td>This shows us as ( x ) gets closer and closer to 0, our y-values decrease rapidly.</td>
</tr>
</tbody>
</table>
- **Key features of the graph:**
  - \( f(x) = \log x \) has a vertical asymptote at \( x = 0 \) (the y-axis).
  - Negative x-values cannot be evaluated in the function \( f(x) = \log x \). They do not exist.
  - The domain of a logarithmic function is \((0, \infty)\).
  - The range of a logarithmic function is \((-\infty, \infty)\).

### Applications

- **A decibel** can be defined as

  \[
  D = 10 \log \frac{I}{I_T}
  \]

  Where \( D \) is decibel level (dB), \( I \) is the measure of intensity (watts per square meter), and \( I_T \) is the intensity threshold of the least audible sound a human is able to hear. In further problems, we will use \( I_T = 1 \times 10^{-12} \frac{W}{m^2} \).

- **Example 1:** Calculate the decibel level associated with the typical sound intensity of a rock band playing with intensity of \( I = 1 \times 10^{-1} \).

  \[
  D = 10 \log \frac{I}{I_T}
  \]

  \[
  D = 10 \log \frac{1 \times 10^{-1}}{1 \times 10^{-12}}
  \]

  \[
  D = 10 \log(1 \times 10^{11})
  \]

  \[
  D = 10 \cdot 11
  \]

  \[
  D = 110 \frac{W}{m^2}
  \]

- **The Richter scale** is used to determine the magnitude of an earthquake. Its equation is given by:

  \[
  M = \frac{2}{3} \log \frac{E}{E_0}
  \]

  Where \( M \) is the magnitude, \( E \) is the seismic energy released by the earthquake (in joules) and \( E_0 \) is the energy released by a reference earthquake \( (E_0 = 10^{4.4} \text{ joules}) \).

- **Example 2:** 1.23 Using the Richter scale, what is the magnitude of an earthquake that released \( 1.27 \times 10^{15} \text{ joules} \) of seismic energy.
\[ M = \frac{2}{3} \log \frac{E}{E_0} \]

\[ M = \frac{2}{3} \log \frac{1.27 \times 10^{15}}{10^{4.4}} \]

\[ M = \frac{2}{3} \log (1.27 \times 10^{10.6}) \]

\[ M \approx \frac{2}{3} (10.704) \]

\[ M \approx 7.136 \]

---

### Properties of Logarithms

- If \( b, M, \) and \( N \) are positive real numbers, where \( b \neq 1 \) and \( p \) and \( x \) are real numbers, then the following are true:
  1. \( \log_b 1 = 0 \)
  2. \( \log_b b = 1 \)
  3. \( \log_b b^x = x \)
  4. \( b^{\log_b x} = x \quad x > 0 \)
  5. **Product Rule**: Log of a product is the sum of the logs.
      \[ \log_b (MN) = \log_b M + \log_b N \]
  6. **Quotient Rule**: Log of a quotient is the difference of the logs.
      \[ \log_b \left( \frac{M}{N} \right) = \log_b M - \log_b N \]
  7. **Power Rule**: Log of a number raised to an exponent is the exponent times the log of the number.
      \[ \log_b M^p = p \log_b M \]

- **Example 1**: Use the properties of logs to simplify the following expressions.
  a) \( \log 1 - \log 1000^x \)

Since \( \log x \) has base 10, we view this expression as \( \log_{10} 1 - \log_{10} 1000^x \). Then use properties 1 and 3 to simplify the expression.

\[ \log 1 - \log 1000^x = 0 - \log(10^3)^x = - \log 10^{3x} = -3x \]

b) \( e^{-3 \ln 2} \)

Use properties of exponents first, then use property 4 of logarithms.
\[
e^{-3\ln 2} = (e^{\ln 2})^{-3} = \frac{1}{(e^{\ln 2})^3} = \frac{1}{2^3} = \frac{1}{8}
\]

c) \( \log_{6} \frac{24}{72} \)

Use the quotient, product, and power rules first, then simplify using properties 2 and 3.

\[
\log_{6} \frac{18}{108} = \log_{6} 18 - \log_{6} 108 = \log_{6} 6 + \log_{6} 3 - \log_{6} 6^2 - \log_{6} 3
\]
\[
= 1 + \log_{6} 3 - 2 \log_{6} 6 - \log_{6} 3 = 1 - 2 + \log_{6} 3 - \log_{6} 3 = -1
\]

- **Example 2:** Write \( \frac{1}{4} \ln(x^2 + 3) - \frac{1}{3} \ln(x^3 - 5) + \ln(x + 2) \) as a single logarithm.

Use the power property on the first and second terms.

\[
= \ln(x^2 + 3)^{1/4} - \ln(x^3 - 5)^{1/3} + \ln(x + 2)
\]

Use the quotient property on the first and second terms.

\[
= \ln \left(\frac{x^2 + 3}{x^3 - 5}\right)^{1/3} + \ln(x + 2)
\]

Use the product property.

\[
= \ln \left[\frac{(x + 2)(x^2 + 3)^{1/4}}{(x^3 - 5)^{1/3}}\right]
\]

- **Example 3:** Write \( \log \left[ \frac{x^2 + 8x - 9}{x^2 - 4x - 12} \right] \) as the sum or difference of logarithms.

Factor the numerator and denominator.

\[
= \log \left[ \frac{(x + 9)(x - 1)}{(x + 2)(x - 6)} \right]
\]

Use the quotient property.

\[
= \log[(x + 9)(x - 1)] - \log[(x + 2)(x - 6)]
\]

Use the product property.

\[
= \log(x + 9) + \log(x - 1) - [\log(x + 2) + \log(x - 6)]
\]

Eliminate brackets.

\[
= \log(x + 9) + \log(x - 1) - (\log(x + 2) - \log(x - 6))
\]

### Change-of-Base Formula

- For any logarithmic bases \( a \) and \( b \) and any positive number \( M \), the change-of-base formula says that

\[
\log_{b} M = \frac{\log_{a} M}{\log_{a} b}
\]

- **Example 1:** Use the change-of-base formula to evaluate \( \log_{5} 26 \). Round to four decimal places.

Use the change-of-base formula

where \( a = 10 \).

\[
\log_{5} 26 = \frac{\log 26}{\log 5}
\]

Approximate with a calculator

\[
\approx 2.024369199
\]

\[
\approx 2.0243
\]
Example 2: Use the change-of-base formula to evaluate \( \log_{\pi} e \). Round to four decimal places.

Use the change-of-base formula where \( a = e \).

\[
\log_{\pi} e = \frac{\ln e}{\ln \pi} = \frac{1}{\ln \pi}
\]

Approximate with a calculator

\[ \approx 0.8735685268 \]

\[ \approx 0.873 \]