Probability

Key Definitions

- **Experiment:** An experiment is a repeatable process where results are observed and recorded (flipping a fair coin).
- **Outcome:** The result of an experiment is called the outcome (the possible outcomes of flipping a coin are heads and tails).
- **Sample Space:** The sample space is the set of a possible outcomes of an experiment, denoted S (the sample space for flipping a coin is heads and tails).
- **Event:** An event is any subset of the sample space, denoted E. Events are typically a single possible outcome of the sample space. A certain event is equal to the sample space and an impossible event is equal to the empty set (the coin landing on heads is an event).
- **Probability of an Event:** The probability of an event within an experiment with equally likely outcomes is the number of items in the event divided by the number of items in the sample space, denoted $P(E)$ and represented as
$$p(E) = \frac{n(E)}{n(S)}$$
(the probability of a coin landing on heads is $\frac{1}{2}$). Event probability is often phrased in terms of successes (achieving a head) over total possible outcomes.
- **Odds of Event:** The odds of an event within an experiment with equally likely outcomes is the number of items in the event compared (in a ratio) to the items in the complement of the event, denoted $o(E)$ and represented as
$$o(E) = n(E): n(E')$$
(the odds of a coin landing on heads is $1:1$). Event odds are often phrased in terms of successes (heads) compared to failures (tails).
- **Expected Value:** The expected value is the estimated long term average of the outcomes of an experiment. Multiply the value of each outcome by its probability and add the results. Note that this is most useful where the probability of different outcomes varies and the event's value has significance (find the best bets to make to make the most money in the long run when the payout of bets varies based on the probability of a success).
Example: A carnival game is designed so that each prize has a certain value and probability. It costs two dollars to play this game and the game company wants to make an average of one dollar per play. The following chart provides the probabilities of each prize and the value of the second and third prizes. Find the value of a first-place prize.

Prize	Prize Cost	Probability
1st	?	.10
2nd	$1.00	.20
3rd	$.50	.30

EV=Probability×(Play Cost − Prize Cost) for each prize where EV=1 and there is a .40 probability that you win nothing.

$$EV = .10(2 - x) + .20(2 - 1) + .30(2 - .5) + .40(2) = 1$$
$$x = 6.5$$

The first place prize costs $6.50.

Rules of Probability

- ***Law of Large Numbers***: If an experiment is repeated enough times, the rate of success of an outcome will approximate the probability of that outcome (flip enough coins and you will achieve a head about half the time).

Rules of Probability		
Rule	Symbols	Definition
1	$p(\emptyset) = 0$	The probability of the impossible event is zero.
2	$p(S) = 1$	The probability of the sample space is one.
3	$0 \leq p(E) \leq 1$	All probabilities are between zero and one.
4	$p(E \cup F) = p(E) + p(F) - p(E \cap F)$	The probability of one of two events is each of their probabilities with their intersection removed.
5	$p(E \cup F) = p(E) + p(F)$ if E and F are mutually exclusive	The intersection of mutually exclusive events is zero.
6	$p(E) + p(E') = 1$	The probability of an event & its complement is one.

Applications

- ***How to Write a Probability***: Probabilities may be written as a fraction, decimal, or percent. Probabilities with small numbers are best understood as fractions while larger numbers and irreducible fractions are best understood using decimals and percentages. Be sure to use the form you are asked for in the question.
- ***Types of Examples***: Common examples of probability where outcomes are equally likely are fair coins and a six-sided die. Common examples using combinatorics are a 52-card deck and lottery games. Expected value refers frequently to examples where outcomes have different probabilities. Conditional probability likes to use polling as the primary example.
- ***Probabilities using Combinatorics***: Use the rules of Counting to decide which method to use. If the experiment is done with replacement, use the Fundamental Principle of Counting (when flipping a coin, you can achieve multiple heads). If the experiment is done without replacement but with order, then use permutations (choosing names out of a hat to win first, second, and third prize). If the experiment is done without replacement and without

order, use combinations (choosing a pair of red socks when picking one at a time out of a drawer of 10 socks, where each pair is a different color).
Example: Find the probability of winning second prize at a lottery that draws 5 balls out of 35.
Second prize means that you picked 4 out of 5 correct numbers. In a lottery, the order of these numbers does not matter. Use combinations to find the probability. First, find all possible combinations of choosing 4 out of 5 right numbers and one wrong number ($n(E)$), then find all combinations of the sample space ($n(S)$). Then, find the probability using the formula above.

$$n(S) = {}_{35}C_5 = \frac{35!}{(35-5)!\cdot 5!} = 324{,}632$$

$$n(E) = {}_5C_4 \cdot {}_{30}C_1 = \frac{5!}{(5-4)!\cdot 4!} \cdot \frac{30!}{(30-1)!\cdot 1!} = 5 \cdot 30 = 150$$

$$\frac{n(E)}{n(S)} = \frac{150}{324{,}632} = .0004621$$

There is approximately a .0005 probability of winning second prize in this lottery. For help with combinations and factorials, see our Counting handout.

- ***Conditional Probability***: The conditional probability of an event, given that it is part of another event is the size of the intersection of the events divided by the size of the given event. Using event A and given event B,

$$p(A|B) = \frac{n(A \cap B)}{n(B)}$$

is the conditional probability (the probability that someone answered yes to a survey given their gender).
Example: Regent University conducted a survey asking students what their favorite ice cream flavor was between chocolate, vanilla, and strawberry. They then sorted the results by gender into the chart below. What is the probability that the student responded chocolate given that they are female?

Survey Results	Chocolate	Vanilla	Strawberry
Male	30	15	5
Female	65	20	15

$$p(\text{Chocolate}|\text{Female}) = \frac{n(\text{Chocolate} \cap \text{Female})}{n(\text{Female})} = \frac{65}{100} = .65$$

The survey states that 65% of females prefer chocolate ice cream to vanilla or strawberry.

Symbol Guide

Probability Symbols		
Term	Symbol	Use
Sample Space	S	$p(S)$
Event	E	$p(E)$
Probability	p	See above
Odds	o	$o(E)$
Size	n	$n(E)$
Given	$\vert$	$p(A\vert B)$